Biased preference models for partnership formation

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Abstract. In multi-group epidemiological models with nonrandom mixing between people in the different groups, often artificial constraints have to be imposed in order to satisfy the balance conditions. Based on the model in [9], we construct a simple biased mixing model where the balance conditions are automatically satisfied as a natural consequence of the equations. The model can be applied to situations where biased partnership formation is central and mixing can be between people in different risk, social, economic, ethnic, or geographic groups. After describing the formulation of the model, we discuss the features of the model and preliminarily study sensitivity of some parameters. We describe the discrete model based on ordinary differential equations and generalize to a continuum described by partial differential equations.

AMS Subject Classfication: 34A34, 34D20, 92B05, 92-06, 92D30

1. Introduction

The spread of sexually transmitted diseases (STD's) is complex. It depends on not only the transmission mechanism but also behaviors of individuals involved in the transmission process. One of the determinants of the spread is the way that individuals select their sexual partners. In a mathematical model for the spread of ATD's, it is important to understand and correctly account for the formulation of their partnerships. In modeling partnerships, the partnership formation must satisfy the balance constraints (see, e.g., [1], [2], [3], [4], [7], [8], [10], and [11]). That is, the number of partnerships formed by people in group A with people in group B in a given period of time must equal the number of partnerships formed by people in group B with people in group A. Because of this balance requirement, if the number of sexual partners for each subpopulation is assumed to be given, often artificial constraints are imposed ([5], [6]), or complicated mathematical derivations are needed ([1], [2], [3]).

There are mixing multigroup models where the balance constraints are automatically satisfied. In the models of [12], [14], [15], and [16], the mixing between

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groups is determined by a stochastic migration matrix, where the entries are based on the fraction of individuals from one group that interact with people in another group. The model in [9] is based on assuming that the number of social contacts and the fraction of individuals in one group that are acceptable to someone in another group is known. The rate of making social contacts between two different groups which result in partnerships is symmetric and the balance constraints are automatically satisfied. They further assume that encounters which form partnerships depend on an individual's sexual drive, on the constraints which morals or fear of sexually transmitted diseases place on behavior, and on current partnership status. These complicated assumptions are epidemiologically reasonable, but they make mathematical analysis extremely difficult.

To simplify the mathematical analysis, we modify the model in [9] and reformulate the partnership formulation in a natural way so that the balance constraints are automatically satisfied. We refer to this simplified version as the Biased Preference Model (BPM). We first consider the discrete case where the population is divided into subgroups. The classification of these subgroups can be based on risk levels, age, social behaviors, economic status, ethnic, or geographic positions. We then discuss the basic features of the model in Section 3. A preliminary mathematical analysis is outlined in Section 4. A generalization of the model to a continuum and a brief discussion are given in Section 5. Some final discussion remarks are made in Section 6.

2. The biased preference model formulation

Divide the susceptible and infected populations into K groups, S_i and I_i, $i = 1, \cdots, K$, and consider the simple transmission model system:

$$\begin{cases} \dfrac{dS_i}{dt} = \mu(S_i^0 - S_i) - \lambda_i S_i, \\ \dfrac{dI_i}{dt} = -\nu I_i + \lambda_i S_i, \end{cases} \qquad i = 1, \cdots, K \tag{2.1}$$

where μ is the natural death rate, $1/\nu$ is the mean duration of the incubation period, and λ_i is the rate of infection.

The rate of infection λ_i is a functional of the model variables among which the formation of partnership plays an essential role. Here a partnership is an activity between two people where the infection can be transmitted (e.g. sexual intercourse). We assume people in each group behave the same when selecting a partner, but have biases between groups. In other words, mixing within each group is assumed to be random but there is nonrandom mixing among these groups.

The formation of partnerships is one of the most important factors in modeling sexually transmitted diseases. It depends on the desirability of an active individual, the acceptability of the potential partners, and the availability of these potential partners.

Let α_{ij} be the preference of people in group i to have a partner from j, that is, the fraction of people in group j with whom each individual in group i desires to form a partnership. This describes the desirability of group j to group i. It is also the acceptability of people in group i to have a partner from group j. If an individual from group i encounters an individual from group j, then the conditional probability that a partnership will form is

$$p(i|j) = \alpha_{ij}\alpha_{ji} \equiv q_{ij}. \tag{2.2}$$

The availability of partners from group j is the probability $p_a(j) = \frac{N_j}{N}$, where $N_i = S_i + I_i$ and $N = \sum_i N_i$. Hence, after an encounter of someone from group i with another individual, the probability of a partnership forming between individuals from group i and group j is

$$p(i,j) = \alpha_{ij}\alpha_{ji}\frac{N_j}{N} = q_{ij}\frac{N_j}{N}. \tag{2.3}$$

We assume that the probability of transmission from an infected partner to a susceptible individual is group independent and we denote it by β. The infection rate of people in group i is

$$\lambda_i = c\beta\sum_{j=1}^{K} p(i,j)\frac{I_j}{N_j} = c\beta\sum_{j=1}^{K} q_{ij}\frac{I_j}{N}, \tag{2.4}$$

where c is the number of encounters per person per unit time, and $\frac{I_j}{N_j}$ is the probability that a person encountered from group j is infected. Here we have assumed that the encounter rate is group independent. The preferences need not be symmetric (i.e. α_{ij} is not necessarily equal to α_{ji}, when $i \neq j$), but the probability of a partnership forming is symmetric since $q_{ij} = \alpha_{ij}\alpha_{ji}$ implies $q_{ij} = q_{ji}$. Also, we note that there is no constraint on $\sum_j \alpha_{ij}$, which may be less than or greater than one.

The model (2.1) with infection rate (2.4) is in a very general setting. Two extreme cases can be easily obtained as follows. $\alpha_{ij} = 0$ and hence $q_{ij} = 0$, $i \neq j$, gives the restricted mixing. $\alpha_{ij} \equiv \alpha_i$, for $j = 1, \cdots, K$, leads to the proportional mixing ([7]).

To simplify the mathematical analysis, we assume that each individual in the population encounter other people at the same rate. If we assume that people in different groups have different rates of encounters, c_i, then the rate of infection can be expressed as

$$\lambda_i = \beta\sum_{j=1}^{K} q_{ij}\frac{c_i I_j}{\sum_k c_k N_k}. \tag{2.5}$$

3. Features of the model

Balance constraints. We denote the number of partners of people in group i from group j by T_{ij}. Note that $T_{ij} = T_{ji}$. In many biased mixing models where an attempt is made to directly control the number of partneced by constructing preferred, selective, or structured mixing functions (see [5], [6], [7], [8], and [13]). However, in the BPM, the balance constraint

$$T_{ij} = cq_{ij}\frac{N_j}{N}N_i = cq_{ji}\frac{N_i}{N}N_j = T_{ji} \tag{3.1}$$

is automatically satisfied. Thus, by using the acceptability α_{ij} or desirability α_{ji} of an individual from group i to an individual from group j as the primary control variable in the biased preference mixing model (instead of the number of partners an individual from group i desires from group j), the balance constraints become a natural consequence of the model, rather than an artificially imposed constraint.

When the rates of encounters in different groups are different, the balance constraint is still satisfied since the total number of partners of people in group i from group j is $T_{ij} = c_i q_{ij}\frac{c_j N_j}{\sum\limits_k c_k N_k} N_i$ which is equal to the number of partners of people in group j from group i, T_{ji}.

The number of partners. The number of sexual partners per individual in many multi-group models is assumed to be constant. When all α_{ij}'s are equal (proportional mixing), this is also true from (2.1) for BPM. However, if the mixing is biased, the number of partners will vary in time depending on the combination of desirability, acceptability, and availability.

From Section 2, the number of partners per person in group i is

$$r_i = c\left(\sum_{j=1}^{K} q_{ij}\frac{N_j}{N}\right), \tag{3.2}$$

which reaches its maximum for the proportional mixing, where $q_{ij} \equiv 1$ (i.e. everyone is equally acceptable as a partner) and $r_i = c$.

If the mixing is biased, people have preferences when choosing their partners. Because the acceptability and the availability must be taken into a consideration and a limitation may occur, $q_{ij} < 1$, and hence $r_i < c$.

Example 3.1. Consider a two group model governed by

$$\begin{cases} \dfrac{dS_i}{dt} = \mu(S_i^0 - S_i) - \lambda_i S_i, \\ \dfrac{dI_i}{dt} = -\nu I_i + \lambda_i S_i, \end{cases} \qquad i = 1, 2, \tag{3.3}$$

with

$$\lambda_i = \frac{c\beta}{N}\left(q_{i1}I_1 + q_{i2}I_2\right).$$

Then $r_i = \frac{c}{N}\left(q_{i1}N_1 + q_{i2}N_2\right)$ and

$$r_1 - r_2 = \frac{c}{N}\left((q_{11} - q)N_1 + (q - q_{22})N_2\right),$$

where $q \equiv q_{12} = q_{21}$. If $q_{11} < q < q_{22}$ or $q_{11} > q > q_{22}$, then r_1 is always less than or greater than r_2 respectively. Otherwise, they may alternate at different times.

We use the following model parameters:

$$\begin{array}{llllll} S_1^0 = 350, & S_1(0) = 350, & I_1(0) = 10, & S_2^0 = 100, & S_2(0) = 100, & I_2(0) = 250, \\ c = 5, & \mu = 0.015, & \alpha_{11} = 0.6, & \alpha_{12} = 1 & \alpha_{21} = 0.5, & \alpha_{22} = 0.2. \end{array}$$

Depending on the probability of transmission, the disease may spread in the population or die out eventually. For example, when $\beta = 0.05$, the disease dies out (see Fig. 1), but when $\beta = 0.1$, the disease persists (see Fig. 2).

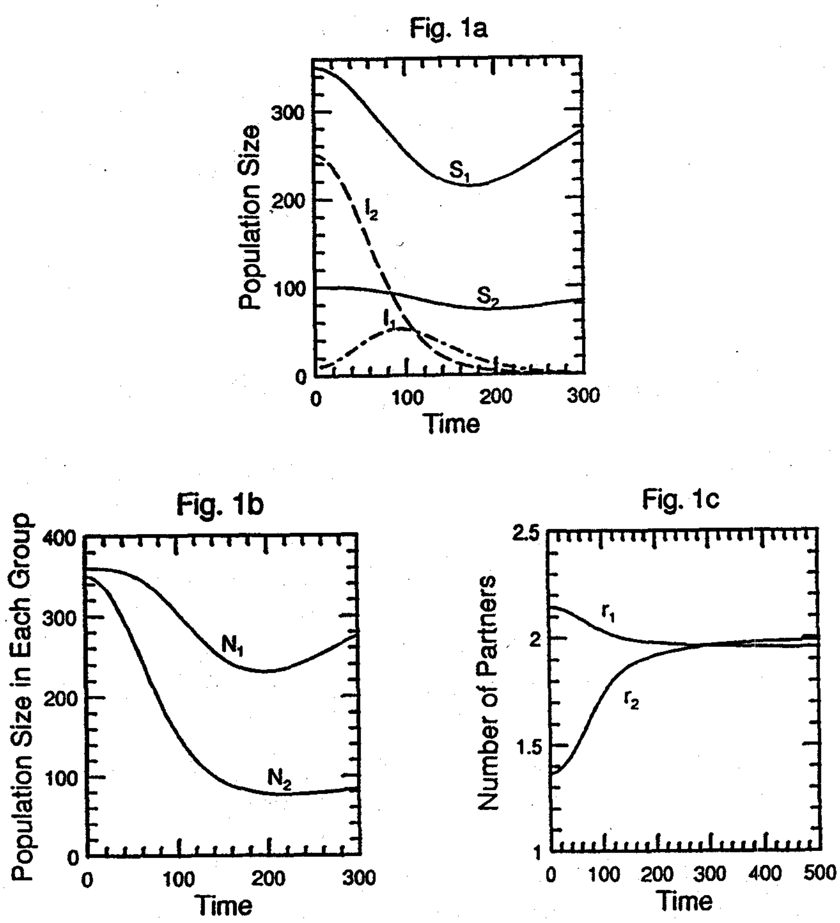

Figure 1

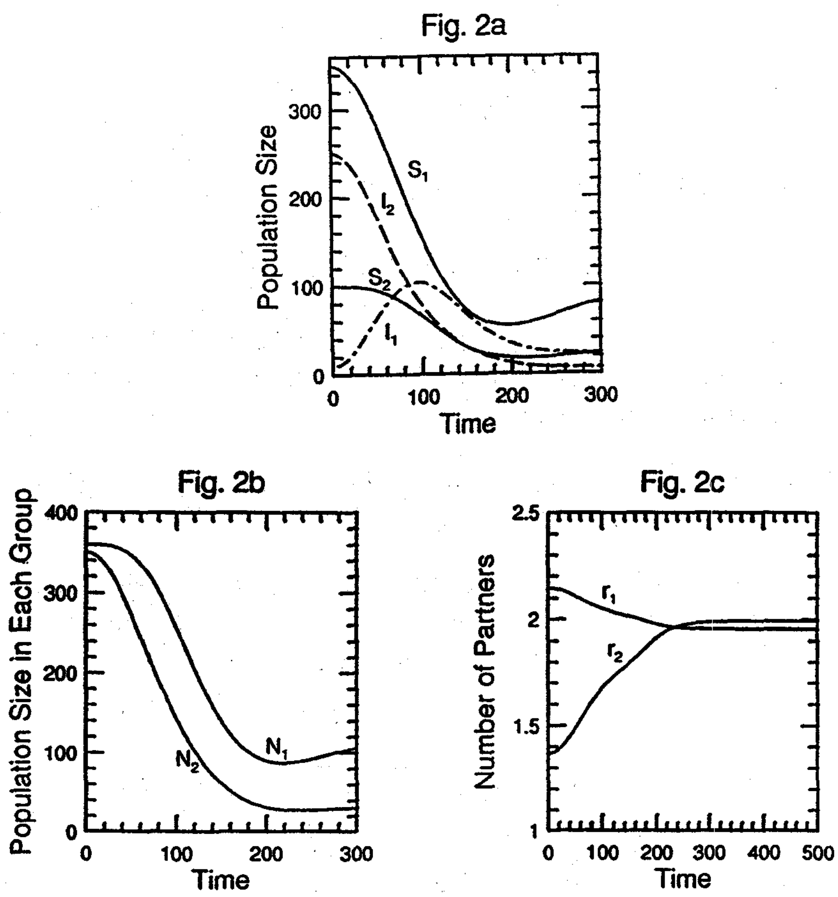

Figure 2

4. Preliminary analysis of the BPM

4.1. Threshold conditions.

The concept of threshold conditions is one of the most important concepts in mathematical epidemiology. It specifies when the disease spreads if a small number of infected people are introduced into the susceptible population. The threshold conditions are usually characterized by the reproductive number which can be obtained by the study of stability of the infection-free equilibrium.

In model (2.1), there is an infection-free equilibrium $\big(S_i = S_i^0, I_i = 0\big)$, $i = 1, \cdots, K$. The stability of this equilibrium is completely determined by the equations of I_i about the equilibrium $I_i = 0$, and can be investigated by either constructing a Liapunov function or examining the eigenvalues of the Jacobian matrix evaluated at the equilibrium.

It is difficult to construct a suitable Liapunov function (see [17]) to determine the reproductive number for the whole population. Hence, we analyze the eigenvalues of the Jacobian matrix of the BPM evaluated at the infection-free equilibrium.

The Jacobian matrix at $I_i = 0$ has the following form of

$$\mathbf{J} = \begin{pmatrix} c\beta q_{11}\frac{S_1^0}{N^0} - \nu & c\beta q_{12}\frac{S_1^0}{N^0} & \cdots & c\beta q_{1K}\frac{S_1^0}{N^0} \\ c\beta q_{21}\frac{S_2^0}{N^0} & c\beta q_{22}\frac{S_2^0}{N^0} - \nu & \cdots & c\beta q_{2K}\frac{S_2^0}{N^0} \\ \vdots & \vdots & \ddots & \vdots \\ c\beta q_{K1}\frac{S_K^0}{N^0} & c\beta q_{K2}\frac{S_K^0}{N^0} & \cdots & c\beta q_{KK}\frac{S_K^0}{N^0} - \nu \end{pmatrix}. \tag{4.1.1}$$

Stability of this Jacobian matrix $\mathbf{J}$ gives threshold conditions for the epidemic.

In general, it is difficult to derive an explicit formula of the reproductive number and usually the eigenvalues of (4.1.1) must be determined numerically. We have investigated this problem analytically for the two-group model (3.3), where the classification of groups may be social, economic, ethnic, or geographic by locating the eigenvalues of the Jacobian matrix at the infection-free equilibrium.

Theorem 4.1.1. *Define the reproductive number by*

$$R_0 = \frac{c\beta}{2\nu N^0}\left(q_{11}S_1^0 + q_{22}S_2^0 + \sqrt{\left(q_{11}S_1^0 - q_{22}S_2^0\right)^2 + 4q_{12}q_{21}S_1^0S_2^0}\right). \tag{4.1.2}$$

Then, if $R_0 > 1$ the epidemic spreads in the population and if $R_0 < 1$ the epidemic dies out.

4.2. Sensitivity studies.

The complex dynamics of the BPM is sensitive to combinations of the model parameters. If the transmission probability β increases or the mean duration of the incubation period $1/\nu$ increases, R_0 increases and the epidemic spreads more rapidly.

If the probability of partnership formation, q_{ij}, increases, more sexual partnerships are formed and the reproductive number increases.

The sexual behavior of individuals is characterized by α_{ij}. For someone in group i, the larger $\sum_j \alpha_{ij}$ is, the less selective they are about whom they form a partnership with and the more partners they will have.

Consider the two group model where the behavior of people in group 2, $(\alpha_{21}, \alpha_{22})$, and the average acceptability of people in group 1, $a \equiv \alpha_{11} + \alpha_{12}$ are fixed. We now use $\alpha_{12} \equiv \alpha$, $0 \le \alpha \le a$, as a parameter to study the effects of the relative acceptability of people in group 2 on the reproductive number. A larger α implies that people in group 1 prefer their partners more from group 2 and are less interested in forming partners within their own group.

In terms of α,

$$R_0(\alpha) = \frac{c\beta}{2\nu N}\left((a-\alpha)^2S_1 + \alpha_{22}^2S_2 + \sqrt{\left((a-\alpha)^2S_1 - \alpha_{22}^2S_2\right)^2 + 4\alpha_{21}^2S_1S_2\alpha^2}\right). \tag{4.2.1}$$

By analyzing R_0 as a function of α, we have the following result.

Theorem 4.2.1. *Assume that preferences in group* 2 *are constant and that* $\alpha_{11} + \alpha = a$ *is fixed, but* α *varies from* 0 *to* a. *Then*

i) *if* $S_1 a^2 \leq \alpha_{22}^2 S_2$, *the reproductive number* $R_0(\alpha)$ *is an increasing function of* α;

ii) *if* $S_1 a^2 > \alpha_{22}^2 S_2$, *there exists a unique* α^*, *in* $(0, a)$, *such that the reproductive number* $R_0(\alpha)$ *assumes its minimum at* α^* *and* $R_0(\alpha)$ *is decreasing as* α *increases from* 0 *to* α^* *and is increasing as* α *increases from* α^* *to* a.

Example 4.2.2. For the two group model (3.2.4) with the parameters

$$S_1^0 = 100, \quad S_2^0 = 200, \quad \beta = 0.2, \quad \mu = 0.015, \quad c = 5, \quad \alpha_{21} = 0.7, \quad a = 1,$$

we let α_{22} increase from 0.3 to 0.75. In Fig. 3 we see that the reproductive number as a function of α is concave. The reproductive number is an increasing function of α when $\alpha_{22} \geq 0.7$ because hypothesis i) in Theorem 4.2.1 is satisfied. When $\alpha_{22} < 0.7$, hypothesis ii) is satisfied and the reproductive number is decreasing. The minimum point α^* increases as α_{22} decreases. The dynamics of the susceptibles and infecteds for different α's are shown in Fig. 4.

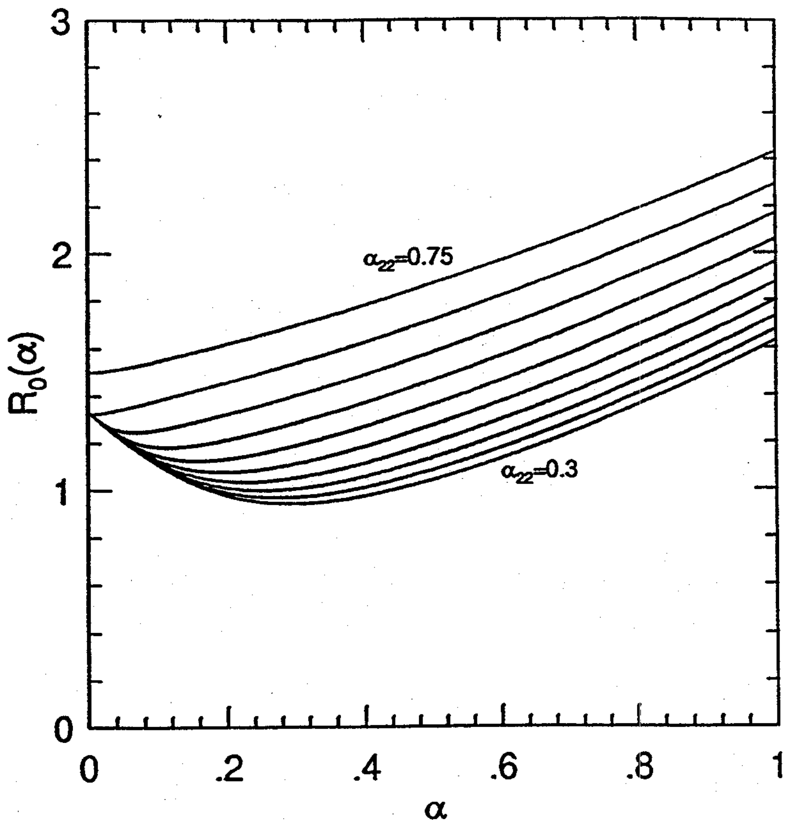

Figure 3

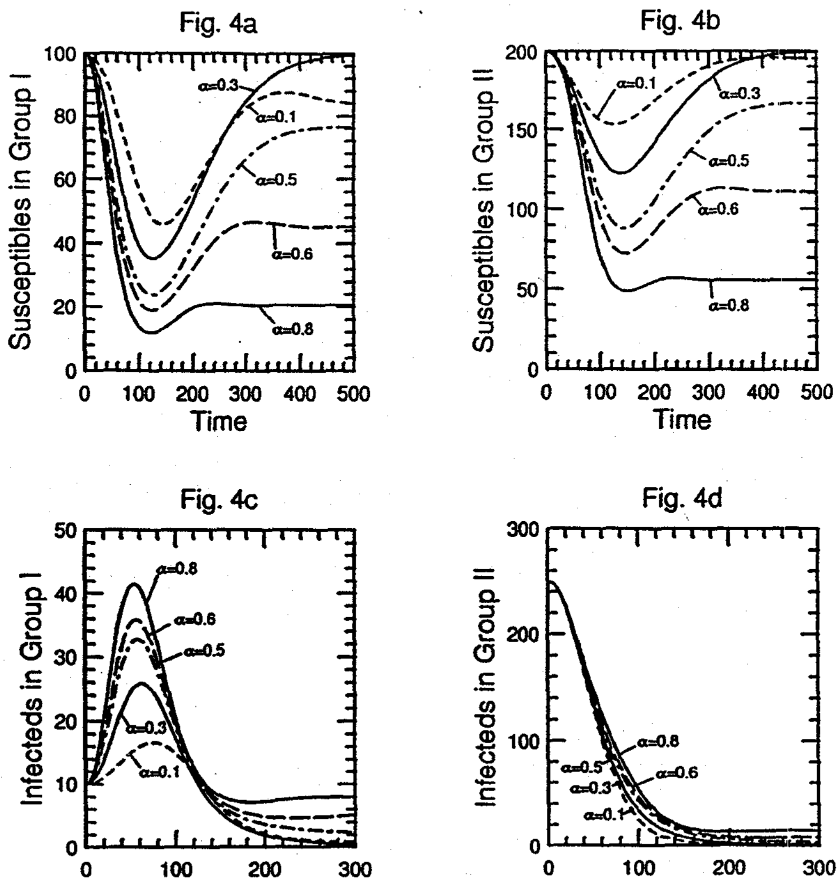

Figure 4

5. Generalization to a continuum

The biased preference model can also be applied to a population with a continuum of biased mixing behavior.

Let x be a continuous state vector of characteristics of individuals in the population (such as age, geographical positions, or behavioral traits) defined in a set X. Let $S(t,x)$ and $I(t,x)$ be continuous densities of the susceptibles and infecteds respectively and $\rho(t,x) = S(t,x) + I(t,x)$.

Assume that the desirability of an individual of state x to form a partnership with an individual of state y is described by $\alpha(x,y)$, $x, y \in X$. Then the acceptability of an individual of state y to an individual of state y is $\alpha(y,x)$.

The availability of individuals with state y in the population is $\rho(t,y)/N(t)$ where $N(t) = \int_{x\in X} \rho(t,x)dx$ is the total population.

Define c as the average number of encounters of each individual in the population when every individual selects their partners randomly. Then the number of partners of an individual of state x per unit time is

$$r(t,x) = \frac{c}{N(t)} \int\limits_{y\in X} \alpha(x,y)\alpha(y,x)\rho(t,y)dy. \tag{5.1}$$

The infection rate of a susceptible individual of state x infected from an infected individual of state y at time t can be expressed by

$$\lambda(t,x) = \frac{c}{N(t)} \int\limits_{y\in X} \beta(x,y)\alpha_(x,y)\alpha(y,x)I(t,y)dy, \tag{5.2}$$

where $\beta(x,y)$ is the transmission rate of the disease from an infected individual of state y to a susceptible individual of state x.

The probability of a partnership forming $q(x,y) \equiv \alpha(x,y)\alpha(y,x)$ is symmetric and the balance constraints are automatically satisfied.

The dynamics of the epidemic is governed by the following system:

$$\begin{aligned} D_tS(t,x) =& \Lambda(x) - (\mu(x) + \lambda(t,x))S(t,x), \\ D_tI(t,x) =& \lambda(t,x) - \nu(x)I(t,x), \end{aligned} \tag{5.3}$$

where D_t denotes the total derivative with respect to time. The threshold conditions are the continuum analogues of those in Section 4.

6. Discussion

We start with the preference behaviors of individuals in the sexually active population to formulate the biased preference models. One of the main features of the biased preference model is that the balance constraints for biased mixing functions are automatically satisfied. The approach can be applied to situations such as mixing between people in different social, economic, ethnic, or geographic groups, where biased partnership formation is central and where the satisfaction of the balance conditions may not be a trivial routine.

The other important feature of the PBM is that the partnerships formed in the population depends on the desirability and acceptability of individuals in each group or each state. Hence, although the number of encounters in each group is fixed, only those encounters which are mutually acceptable result in partnerships. We believe that this is more reasonable than assuming the number of partners in each group is fixed. Moreover, this important feature gives more flexibility in modeling nonrandom mixing to include other factors for the spread of the disease as is discussed in [12].

For mathematical simplicity, we assume that the desirability and the acceptability are constant in this article, which means that people do not change their acceptability no matter what happens in the environment. More realistically, people would adjust their acceptability according to availability of their desired partners. Hence, the acceptabilities α should be density dependent. That is, they are functional of N_i, which certainly increases difficulty in mathematical analysis.

To consider availability, we assume the probability to encounter a person from group j is N_j/N or $c_jN_j/\sum\limits_k c_kN_k$. This is based on the homogeneous mixing assumption that people in the population have an equal probability to select their partners. This is not valid for heterogeneous populations such as a two sex population. In order to model those populations, the availability formulation needs to be modified.

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